

Entropy Bound Derived from the New Thermodynamics on Holographic Screen

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Abstract

We introduce a new interpretation of chemical potential and show that holographic entropy is entropy bound, which is supported by two ideal cases discussed in detail. One is sparse but incompressible liquid like a star of uniform density and the other is a screen at infinity in spherically symmetric spacetime. Our work is based on the new scenario of entropy force and holographic thermodynamics, and the Brown-York quasi-local energy.

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I. INTRODUCTION

In 1970s, the discovery of black hole entropy, the four laws of classical black hole mechanics and Hawking radiation revealed a mysterious but charming connection between gravitation and thermodynamics. In 1995, Jacobson derived the Einstein equations as equilibrium equation of state by assuming proportionality of entropy and horizon area, and the Clausius relation. Since then, a large number of papers have sprung up to generalize the approach to other gravity theories [1][2][3]. Last year, Verlinde [4] argued the gravity is not only an emergent force, but particularly an entropy force. His work is motivated by the Jacobson's work as well as the gauge/gravity correspondence.

Recently, Li and his collaborators gave a new scenario of entropy force and holographic thermodynamics [5]. First, a surface stress tensor is introduced on a holographic screen while Brown-York quasi-local energy [6] is adopted to replace Verlinde energy. This tensor would give both the surface energy and pressure. Then, a closed screen used by Verlinde is extended to an open or closed one. Finally, a time-like screen is used rather than a null screen. The thermodynamics is self-consistent and back to the Jacobson's form while the time-like screen is approaching the null screen.

As an important part, chemical potential on holographic screen is discussed in detail, and specific examples are calculated in Li's paper. For example, a screen sweeping vacuum outside a black hole horizon gives a form of chemical potential according to thermodynamic relations.

In this paper, we propose a different interpretation of chemical potential on holographic screen. We ponder that such holographic entropy is the entropy bound, independent of concrete matter, since the entropy is given within an ideal system. There are two reasons in favor of our point. One is the Verlinde temperature is general, having nothing to do with the composition of matter. It only requires the information of metric. The other is Birkhoff's theorem ensure the irrelevance between exterior metric and interior matter composition.

In section III, we discuss two ideal cases, each of which will lead to the entropy bound respectively. For the first case, we analyze the system with sparse but incompressible liquid like star of uniform density and then give a holographic derivation of Bekenstein entropy bound. For the second case, we consider a spherically symmetric system, and compute its holographic entropy on a screen at infinity. Since the temperature we use is not related to

specific matter system, the entropy we obtain is not the usual statistical one (like the black hole entropy). At infinity, we also can not recognize the form of system, so the holographic entropy which we calculated is the entropy bound. In this paper, we choose the natural units (ie. $\hbar = c = 1$)

II. PRELIMINARY

In this paper, we need the form of the surface stress tensor and other indispensable quantities, thus we have to review [5][7], and lay the foundation of our calculation in section III.

Introducing some basic definitions is beneficial. We set a 2+1 dimensional timelike hypersurface Σ embedded in the 3+1 dimensional spacetime M . x^a , g_{ab} , ∇_a , y_i , γ_{ij} and D_i (here a, b run from 0 to 3, and i, j run from 0 to 2) denote the coordinates, metric and covariant derivatives of M and Σ respectively. The hypersurface Σ can be described by a scalar function

$$f_\Sigma(x^a) = c. \quad (1)$$

Analogous to Jacobson's procedure [8], we suppose an energy flux δE passing through an open but time-like patch on holographic screen $d\Sigma = dAdt$

$$\delta E = \int_\Sigma T_{ab} \xi^a N^b dAdt, \quad (2)$$

where T_{ab} is the stress tensor in the bulk M , ξ^a is the Killing vector, and N^a is the unit vector normal to Σ . Obviously, a vector normal to Σ is proportional to $g^{ab} \nabla_b f$, and can be normalized as

$$N^a = \frac{g^{ab} \nabla_b f_\Sigma}{\sqrt{\nabla_b f_\Sigma \nabla^b f_\Sigma}}. \quad (3)$$

From the viewpoint of the holographic principle, every physical process in the bulk is corresponding to a process on the holographic screen. When an energy flux δE passes through the screen Σ from the left side to the right side, the energy will change on the right side, and thus a corresponding change on the screen due to the holographic principle. The energy density ε and energy flux j on Σ take the form

$$\varepsilon = u_i \tau^{ij} \xi_j, \quad j = -m_i \tau^{ij} \xi_j, \quad (4)$$

where τ^{ij} is the energy-momentum tensor on Σ , u_i and m_i are the unit vectors normal to the screen's boundary $\partial\Sigma$ (we will give the expressions of u_i and m_i below Eq.(8)), and ξ_i is the Killing vector on Σ . Naturally, τ^{ij} should only rest on the geometry of the boundary, thus we suppose (in [7], a more general and appropriate ansatz has been given, which, however, do not influence our discussion)

$$\tau^{ij} = n(K^{ij} - K\gamma^{ij}), \quad (5)$$

where n is a constant to be determined, and K^{ij} is the extrinsic curvature on Σ defined as

$$K_{ij} = -e_i^a e_j^b \nabla_a N_b. \quad (6)$$

Here $e_i^a = \frac{\partial x^a}{\partial y^i}$ is the projection operator satisfying $N_a e_i^a = 0$. It should be stressed that Eq.(5) is the key ansatz in this scheme to derive the Einstein equations. The total energy variation on the screen then becomes

$$\delta E = (u_i \tau^{ij} \xi_j) dA|_t^{t+dt} - m_i \tau^{ij} \xi_j dy^i dt = - \int_{\partial\Sigma} (M_i) \tau^{ij} \xi_j \sqrt{h} dz^2 = - \int_{\Sigma} D_i \tau^{ij} \xi_j dAdt. \quad (7)$$

Notice that there are two contributions to δE , namely the variation of the energy density ρ and the energy flux j flowing into the screen. h is the determination of the reduced metric on $\partial\Sigma$, with D_i the covariant derivative on Σ and M_i a unit vector in Σ and normal to $\partial\Sigma$. Let us choose a suitable function $f_{\partial\Sigma}(y^i) = c$ on Σ to denote the boundary $\partial\Sigma$, then we have

$$M^i = \frac{\gamma^{ij} D_j f_{\partial\Sigma}}{\sqrt{D_j f_{\partial\Sigma} D^j f_{\partial\Sigma}}}. \quad (8)$$

When M^i is along the direction of $dy^0(dt)$ it becomes u^i , and when it is along the direction of $dy^i(dy^1, dy^2)$ it becomes m^i .

From Eq.(5) and the Gauss-Codazzi equation $(R_{ab} - Rg_{ab}/2)N^a e_i^b = -D_j(K^{ij} - K\gamma^{ij})$, one can rewrite the Eq.(7) as

$$\delta E = \int_{\Sigma} m(R_{ab} - \frac{R}{2}g_{ab}) \xi^i e_i^a N^b dAdt. \quad (9)$$

In the following derivations, we will only focus on the quasi-static process for simplicity. Recalling that we have use Eq.(1) to denote the holographic screen Σ . In the quasi-static limit, $f_{\Sigma}(x^a)$ is independent of time, so $N_a \sim (0, \partial_1 f_{\Sigma}, \partial_2 f_{\Sigma}, \partial_3 f_{\Sigma})$. Note that $\xi^a \simeq (1, 0, 0, 0)$ in the quasi-static limit, thus we have $N_a \xi^a \rightarrow 0$. Then the Killing vector ξ_i on Σ can be induced from the Killing vector ξ_a in the bulk B ,

$$\xi_i = \xi_a e_i^a, \quad D_{(i} \xi_{j)} = \nabla_a (\xi_b - N_c \xi^c N_b) e_{(i}^a e_{j)}^b = K_{ij} N_a \xi^a \rightarrow 0. \quad (10)$$

Thus, in the quasi-static limit, we can rewrite Eq.(2) as

$$\delta E = \int_{\Sigma} T_{ab} \xi^i e_i^a N^b dA dt. \quad (11)$$

From the Eq.(11) and Eq.(9), it follows

$$n(R_{ab} - \frac{R}{2}g_{ab})e_i^a N^b = T_{ab}e_i^a N^b. \quad (12)$$

Note that $g_{ab}e_i^a N^b = 0$, thus we can add a cosmological constant term Λg_{ab} to the left of the above equation. Defining the Newton's constant as $G = \frac{1}{8\pi n}$, we then get the Einstein equations,

$$R_{ab} - \frac{R}{2}g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \quad (13)$$

Substitute $n = \frac{1}{8\pi G}$ into Eq.(5), we obtain,

$$\tau^{ij} = \frac{1}{8\pi G}(K^{ij} - K\gamma^{ij}), \quad (14)$$

which is just the quasi-local stress tensor defined in [6]. As emphasized in [6], τ^{ij} characterizes the entire system, including contributions from both the gravitational field and the matter fields. In addition, generally, τ^{ij} need to be subtracted in order to derive the correct energy.

To find the relationships between the energy in the bulk and on the holographic screen, we require a closed screen. Using Eq.(2) and Eq.(7), together with the conservation of energy in the bulk

$$\int \sqrt{-g} \nabla_a (T^{ab} \xi_b) dx^4 = -(U_a T^{ab} \xi_b) dV|_t^{t+dt} + \int_{\Sigma} T_{ab} \xi^a N^b dA dt = 0, \quad (15)$$

we get

$$(u_i \tau^{ij} \xi_j) dA|_t^{t+dt} = (U_a T^{ab} \xi_b) dV|_t^{t+dt}, \quad (16)$$

where the fact that a closed screen has no time-like boundary has been used. This shows the relationship between the bulk energy and the energy on the holographic screen, which also make the logic in section III complete. Obviously, they differ from each other by a time independent constant at most. This constant is associated with the subtraction scheme of the energy on the screen, and generally it depends on the position of the screen. Due to the constant independent of time, the subtraction scheme does not affect our derivation of the Einstein equations.

III. CALCULATION

In this section, we follow the procedure in Sec. VI of [6], and delve into the first law of thermodynamics on the holographic screen.

Let us consider a 2+1 dimensional holographic screen Σ with a constant time slice B . For simplicity, we will deal with the case of a static, spherically symmetric spacetime

$$ds^2 = -N^2 dt^2 + h^2 dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (17)$$

where N and h are functions of r only, and Σ is a hypersurface described by $r = R = \text{constant}$. Mass is distributed around the origin point with spherical symmetry. We use the quasi-local energy in [6] to investigate the first law of thermodynamics on screen.

As mentioned in Sec. II, an appropriate subtraction scheme for the stress tensor on Σ is necessary to derive the correct energy. For the asymptotically flat space, Brown and York propose to deduct the contributions of energy from the flat background spacetime, which gives rise to the same result at infinity as the ADM energy for asymptotically flat space. Thus, we concentrate on spherically symmetric asymptotically flat spacetime in this section. More details of the subtraction scheme available from Sec. VI of [6], we merely use the results without proof.

We choose Newton constant $G = 1$ for simplicity. Following the subtraction scheme used in [6], the proper surface energy density on screen becomes

$$\varepsilon = u_i \xi_j \tau^{ij} = \frac{N}{4\pi} \left(\frac{1}{r} - \frac{1}{rh} \right) \Big|_R, \quad (18)$$

while the proper surface momentum density j_a vanishes due to spherical symmetry. And the pressure p , trace of the spatial stress, is

$$p = \frac{1}{2} \sigma_{\dot{a}\dot{b}} \tau^{\dot{a}\dot{b}} = \frac{1}{8\pi} \left(\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r} \right) \Big|_R. \quad (19)$$

u_i is the unit vector normal to boundary $\partial\Sigma$, and ξ_i is the Killing vector on Σ . $\sigma_{\dot{a}\dot{b}}$ is the induced metric on two-boundary B . Both ε and p are measured by observer at infinity and vanish in flat spacetime. Thus, the quasi-local energy on a time slice B is

$$E = \int_B d^2x \sqrt{\sigma} \varepsilon = N \left(r - \frac{r}{h} \right) \Big|_R. \quad (20)$$

Introduce the Verlinde temperature on the screen as

$$T = \frac{1}{2\pi} e^\phi N^b \nabla_b \phi = \frac{1}{2\pi} \frac{N'}{h}. \quad (21)$$

Then, we assume that the full form of the first law of thermodynamics on screen is

$$dE = TdS - pdA + \mu dN_f, \quad (22)$$

where S is the entropy of screen, μ is the chemical potential and N_f is the number of freedom. Here we suppose $N_f = A$ as in Verlinde's paper.

We will discuss an ideal model in which mass density ρ is a sufficiently small constant independent of r . It suggests incompressible liquid and infinite sound speed like stars of uniform density. Thus, the holographic entropy we calculate is rather the entropy bound of ordinary matter. It will give some intuitions about specific physical system. In GR, with a constant ρ , there are exact solutions of metric [9]:

$$N(r) = \frac{3}{2} \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} - \frac{1}{2} \left(1 - \frac{2Mr^2}{R^3}\right)^{\frac{1}{2}} = \frac{3}{2} \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} - \frac{1}{2} \left(1 - \frac{8\pi\rho r^2}{3}\right)^{\frac{1}{2}}, \quad (23)$$

$$h(r) = \frac{1}{\sqrt{1 - 2\frac{m(r)}{r}}} = \frac{1}{\sqrt{1 - \frac{8\pi\rho r^2}{3}}}, \quad (24)$$

where R is the radius of matter distribution and M is the total mass.

We initially suspend the chemical potential and assume the first law of thermodynamics on the holographic screen

$$dE = TdS - pdA. \quad (25)$$

Using Brown and York's energy definition, relative quantities are as follows:

$$T = \frac{N'}{2\pi h} = \frac{2\rho r}{3}, \quad (26)$$

$$p = \frac{1}{2} \sigma_{\dot{a}\dot{b}} \tau^{\dot{a}\dot{b}} = \frac{1}{8\pi} \left(\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r} \right) = \frac{\rho}{4} \left(1 - \sqrt{1 - \frac{2M}{R}} \right) - \frac{\pi\rho^2}{18} \left(1 + 3\sqrt{1 - \frac{2M}{R}} \right), \quad (27)$$

$$E = \frac{4\pi\rho r^3}{3} \left(\frac{3}{2} \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} + \frac{2\pi\rho r^2}{3} - \frac{1}{2} \right), \quad (28)$$

thus we can get

$$S = 3\pi \sqrt{1 - \frac{2M}{R}} r^2 + \frac{2\pi^2 \rho r^4}{3} \left(1 + 3\sqrt{1 - \frac{2M}{R}} \right) = 3\pi \sqrt{1 - \frac{2M}{R}} r^2 + \frac{8\pi^2 \rho r^4}{3} + o(\rho^2). \quad (29)$$

When $\rho \Rightarrow 0$, we have $S = 3\pi r^2$. Actually, without matter, no normal entropy appears. Thus, we introduce a holographic chemical potential, and then the 1st law is back to the form of Eq.(22). Then, we immediately have

$$\mu = \frac{\sqrt{1 - \frac{2M}{R}}\rho r}{2} = \frac{\rho r}{2} + o(\rho^2) = \frac{\rho r}{2} = \frac{3T}{4}. \quad (30)$$

This is proportional to T and ρr , which denotes the influence of particles passing through the screen. Finally, we get the entropy bound $S = \frac{8\pi^2 \rho r^4}{3} \approx 2\pi E r$ rather than the entropy of concrete matter.

As the second model, we place a screen sweeping vacuum at infinity in the spherically symmetric space. According to Birkhoff's theorem, no matter what the composition of matter is and how it distributes (up to spherical symmetry) and evolves, the geometry of outer space-time is described by Schwarzschild vacuum solution, such as black hole, star system. This is very similar to the case with constant bulk density. Again, the specific mathematical calculation of entropy is just entropy bound of some special physical system.

Still without the chemical potential like Eq.(25), those quantities have different forms:

$$T = \frac{N'}{2\pi h} = \frac{M}{2\pi r^2}, \quad (31)$$

$$p = \frac{1}{2}\sigma_{\dot{a}\dot{b}}\tau^{\dot{a}\dot{b}} = \frac{(1 - \sqrt{1 - \frac{2M}{r}})r - M}{8\pi r^2}, \quad (32)$$

$$E = Nr(1 - \frac{1}{h}) = 2M + r(\sqrt{1 - \frac{2M}{r}} - 1). \quad (33)$$

The entropy is approximately

$$S \approx \pi(-r^2 + \sqrt{1 - \frac{2M}{r}}r(3M + r) + 3M^2 \log 2(-M + r + \sqrt{1 - \frac{2M}{r}}r)). \quad (34)$$

When $r \Rightarrow \infty$, the mass M can be treat as a small quantity, then we expand S as

$$S \approx 2\pi r M + (\frac{\pi}{2} - 3\pi \log(M) + 3\pi \log(2r))M^2 - \frac{(5\pi M^3)}{r} \approx 2\pi M r. \quad (35)$$

This is the Bekenstein bound. The entropy $S \Rightarrow 0$ while the ADM mass $M \Rightarrow 0$, therefor the holographic chemical potential vanishes.

The reason why we lay the screen at infinity is attributed to the non-locality of gravity. It requires an asymptotic flat spacetime to define the total energy. To include the whole

physics, especially the gravity, we choose an isolated system which have infinite boundary to satisfy the asymptotic flat condition. Only in this way can we obtain the entropy bound of the system. To some extent, this is coincident with two simple calculations in [10] (page 105 and 112) which show that a large r is needed to reach the entropy bound for photons. Those examples correspond to concrete material system and are included in our general calculation.

Let us analyze the mathematical structure of the first law of holographic thermodynamics. In this paper, we just care about the physics of isolated systems. Consequently we consider the ADM mass M and constant density ρ are parameters rather than variables. With the only variable r , the holographic thermodynamic equation is totally differential.

IV. DISCUSSION AND CONCLUSION

On Verlinde and Li's projects, gravity is treated as entropy force which is not related to elementary particles, while the chemical potential is related to the variation of the number of particles in thermodynamics. In the first model, the screen moving outward in the star leads to the increase of particles, therefore the holographic chemical potential appears in the screen, whereas the chemical potential in the second model does not arise owing to the screen moving in the vacuum. Since the specific matter constitution is not considered, the holographic chemical potential cannot be deduced from the first principle in this paper. We can only judge the origin of the chemical potential from simple discussion.

Since the gravity is not fundamental and have no relevant elementary particles, we can relate it to phonon. The degrees of freedom are equal to the area in our convention. While the screen expanding in the vacuum, the degrees of freedom also rise without particles, thus it is contributed by the particles and gravity respectively. The effect of gravity in the holographic screen is the increasing number of modes of simple harmonic oscillation. A larger screen area implies the possibility of excitation of higher energy levels. We know the phonon have no chemical potential, therefore the screen moving in vacuum will not lead a chemical potential as the second case.

Having calculated these two special examples, we find that entropy bound is a natural result of the first law of thermodynamics on the holographic screen. However, more research on holographic chemical potential need to be done in future.

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